KINETICS OF WEAKLY TURBULENT WAVES IN A NONSTATIONARY FLOW

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The kinetic equation for weakly turbulent waviness is obtained and investigated in [1, 2]. Distortion of such waviness by a stationary spatially homogeneous flow is examined in [3]. The perturbation of a linear system of surface waves by a random velocity field with given characteristics is studied in [4]. The influence of a weak nonstationary flow on surface waviness is examined in this paper. An analog of the collision integral [1, 2] as well as components describing the linear and nonlinear system responses, nonlocal in time, to the perturbation are present in the kinetic equation obtained.

The potential motion of a fluid allows of a Hamiltonian description whose Hamiltonian function in the presence of a flow with the velocity $\mathbf{v}(\mathbf{x}, z, t)$ has the form [4]

$$H=\frac{1}{2}\int d\mathbf{x}\int\limits_{-\infty}^{\eta(\mathbf{x})}(\nabla\varphi-\mathbf{v})^2\,dz+\frac{g}{2}\int d\mathbf{x}\eta^2(\mathbf{x}). \tag{1}$$

Here $\varphi(\mathbf{x}, z, t)$ is the hydrodynamic potential, and $z = \eta(\mathbf{x}, t)$ is the equation of the surface. The canonic variables for the Hamiltonian (1) are $\eta(\mathbf{x}, t)$, $\psi(\mathbf{x}, t) = \varphi(\mathbf{x}, z, t)|_{z=\eta}$. In the case of weakly nonlinear, nondecaying wave processes it is convenient to go over to the normal variables $b_{\mathbf{k}}(t), b_{\mathbf{k}}^*(t)$ [1, 2]. Under the weak action of the velocity field $\mathbf{v}$ on an unperturbed system, the first terms of the expansion (1) in powers of $b_{\mathbf{k}}^*$, $b_{\mathbf{k}}$ and $\mathbf{v}$ yield an effective Hamiltonian in the form of the sum of the Hamiltonians $H_2 + H_4$ investigated in [1, 2] and the addition $H_{\mathbf{v}}$ that is linear in $\mathbf{v}$:

$$\begin{gathered}H=H_2+H_4+H_{\mathbf{v}},\quad H_2=\int\omega_{\mathbf{k}}b_{\mathbf{k}}^*b_{\mathbf{k}}d\mathbf{k},\\ H_4=\frac{1}{2}\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}b_{\mathbf{k}}^*b_{\mathbf{k}_1}^*b_{\mathbf{k}_2}b_{\mathbf{k}_3}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3,\\ H_{\mathbf{v}}=\int R_{\mathbf{k}_1\mathbf{k}_2}b_{\mathbf{k}_1}^*b_{\mathbf{k}_2}d\mathbf{k}_1\,d\mathbf{k}_2+\int d\mathbf{k}_1\left[\left(P_{\mathbf{k}_1}b_{\mathbf{k}_1}^*+\int Q_{\mathbf{k}_1\mathbf{k}_2}b_{\mathbf{k}_1}^*b_{\mathbf{k}_2}^*d\mathbf{k}_2\right)+\text{c.c.}\right].\end{gathered} \tag{2}$$

Here $H_{\mathbf{v}}$ describes surface wave scattering by the velocity field $\mathbf{v}$:

$$R_{\mathbf{k}_1\mathbf{k}_2}=\frac{i}{2\pi}\sqrt{k_1k_2}\left(\frac{\omega_2}{\omega_1}\right)^{1/2}\int[u_{\mathbf{k}}^*\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)-u_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2)]\,d\mathbf{k};$$

$$u_{\mathbf{k}}=\int d\mathbf{x}e^{i\mathbf{k}\mathbf{x}}\int\limits_{-\infty}^{0}dze^{kz}\,[i\mathbf{k}\mathbf{v}(\mathbf{x},z,t)+kv_z(\mathbf{x},z,t)].$$

Analogous formulas can be obtained for the functions P and Q while the formula for T is in [5]. The equation of motion for the dynamic variable $f(b^*, b)$ has the form

$$\frac{df}{dt}=\{H,f\}=i\int d\mathbf{k}\left(\frac{\delta H}{\delta b_{\mathbf{k}}}\frac{\delta f}{\delta b_{\mathbf{k}}^*}-\frac{\delta H}{\delta b_{\mathbf{k}}^*}\frac{\delta f}{\delta b_{\mathbf{k}}}\right). \tag{3}$$

For a statistical description of waviness we should go from (3) over to equations for the mean values which are obtained by taking the average of equalities of the type (3) for the sequences of all the more complicating combinations of variables. The closure of these equations by using decoupling of the means results in a kinetic equation of which the most interesting is the equation for the quantities $\langle b_{\mathbf{k}}^*b_{\mathbf{k}'}\rangle=M_{\mathbf{k}\mathbf{k}'}$ associated with the amplitude spectrum of the surface waviness, its nondiagonality in the momentum is due to the presence of a spatially inhomogeneous velocity field $\mathbf{v}$. By taking the average for $f = b_{\mathbf{k}}^*b_{\mathbf{k}}$, we obtain

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$$\frac{d}{dt} M_{\mathbf{k}\mathbf{k}'} = \langle \{H, b^*_{\mathbf{k}} b_{\mathbf{k}'}\}\rangle = (\Omega_2 + R_2) M + T_2 S, \tag{4}$$

where $S_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}'_1\mathbf{k}'_2} = \left\langle b^*_{\mathbf{k}_1} b^*_{\mathbf{k}_2} b_{\mathbf{k}'_1} b_{\mathbf{k}'_2}\right\rangle$, and the operators Ω_2, R_2, and T_2 are generated by the Hamiltonians H_2, $H_{\mathbf{v}}$, and H_4, respectively:

$$(\Omega_2 + R_2) M = i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) M_{\mathbf{k}\mathbf{k}'} + i \int d\mathbf{k}_1 \left(R_{\mathbf{k}\mathbf{k}_1} M_{\mathbf{k}'\mathbf{k}_1} - R_{\mathbf{k}_1\mathbf{k}'} M_{\mathbf{k}_1\mathbf{k}}\right),$$

$$T_2 S = i \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \left[T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} S_{\mathbf{k}'\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) - T_{\mathbf{k}'\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} S_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}\mathbf{k}_1} \delta(\mathbf{k}' + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\right],$$

where the contributions of the terms with the functions P and Q proportional to v^2 are omitted. The quantity $T_2 S = \langle \{H_4, b^*_{\mathbf{k}} b_{\mathbf{k}'}\}\rangle$ is not known in Eq. (4), so it is next necessary to write an equation analogous to (4):

$$\frac{d}{dt} T_2 S = \langle \{H, \{H_4, b^*_{\mathbf{k}} b_{\mathbf{k}'}\}\}\rangle = (\Omega_4 + R_4) T_2 S + T_4 T_2 Y,$$
$$Y_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}'_1\mathbf{k}'_2\mathbf{k}'_3} = \left\langle b^*_{\mathbf{k}_1} b^*_{\mathbf{k}_2} b^*_{\mathbf{k}_3} b_{\mathbf{k}'_1} b_{\mathbf{k}'_2} b_{\mathbf{k}'_3}\right\rangle. \tag{5}$$

The solution of (5) by iteration yields, to the accuracy of terms linear in $\mathbf{v}$,

$$T_2 S = \Big\langle \left(\frac{d}{dt} - \Omega_4\right)^{-1} \{H_4, \{H_4, b^*_{\mathbf{k}} b_{\mathbf{k}'}\}\} + \left(\frac{d}{dt} - \Omega_4\right)^{-1} \times$$
$$\times \left\{H_{\mathbf{v}}, \left(\frac{d}{dt} - \Omega_4\right)^{-1} \{H_4, \{H_{\mathbf{v}}, b^*_{\mathbf{k}} b_{\mathbf{k}'}\}\}\right\}\Big\rangle = \left(\frac{d}{dt} - \Omega_4\right)^{-1}\left[1 + R_4 \left(\frac{d}{dt} - \Omega_4\right)^{-1}\right] T_4 T_2 Y.$$

Considering the correlator Y a slowly varying function of the time, we write this last formula in the form

$$T_2 S = \left[1 + \left(\frac{d}{dt} - \Omega_4\right)^{-1} R_4\right] (-\Omega_4^{-1}) T_4 T_2 Y. \tag{6}$$

We substitute (6) into (4) and following [1] decouple Y into a sum of trilinear terms in M. We consequently obtain the kinetic equation

$$\frac{dM}{dt} = (\Omega_2 + R_2) M + (-\Omega_4^{-1}) T_4 T_2 Y + \left(\frac{d}{dt} - \Omega_4\right)^{-1} R_4 (-\Omega_4^{-1}) T_4 T_2 Y,$$
$$J_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6} = \left(M_{\mathbf{k}_1\mathbf{k}_3} M_{\mathbf{k}_2\mathbf{k}_4} + M_{\mathbf{k}_1\mathbf{k}_4} M_{\mathbf{k}_2\mathbf{k}_3}\right) M_{\mathbf{k}_5\mathbf{k}_6}, \tag{7}$$

in whose right side the second component equals

$$(-\Omega_4)^{-1} T_4 T_2 Y(M) = \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}'_1 d\mathbf{k}'_2 d\mathbf{k}'_3 \times$$
$$\times \left\{\frac{i}{\omega' + \omega_1 - \omega_2 - \omega_3 - i0} T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \times\right.$$
$$\times \left(T_{\mathbf{k}'\mathbf{k}'_1\mathbf{k}'_2\mathbf{k}'_3} \delta(\mathbf{k}' + \mathbf{k}'_1 - \mathbf{k}'_2 - \mathbf{k}'_3) J_{\mathbf{k}_3\mathbf{k}_2\mathbf{k}'_3\mathbf{k}'_2\mathbf{k}'_1\mathbf{k}_1} + T_{\mathbf{k}_1\mathbf{k}'_1\mathbf{k}'_2\mathbf{k}'_3} \delta(\mathbf{k}_1 + \mathbf{k}'_1 - \mathbf{k}'_2 - \mathbf{k}'_3) J_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}'_3\mathbf{k}'_2\mathbf{k}'_1\mathbf{k}} -\right.$$
$$- T_{\mathbf{k}_2\mathbf{k}'_1\mathbf{k}'_2\mathbf{k}'_3} \delta(\mathbf{k}_2 + \mathbf{k}'_1 - \mathbf{k}'_2 - \mathbf{k}'_3) J^*_{\mathbf{k}'\mathbf{k}_1\mathbf{k}'_2\mathbf{k}'_3\mathbf{k}'_1\mathbf{k}_3} -$$
$$\left.\left.- T_{\mathbf{k}_3\mathbf{k}'_1\mathbf{k}'_2\mathbf{k}'_3} \delta(\mathbf{k}_3 + \mathbf{k}'_1 - \mathbf{k}'_2 - \mathbf{k}'_3) J^*_{\mathbf{k}'\mathbf{k}_1\mathbf{k}'_2\mathbf{k}'_3\mathbf{k}'_1\mathbf{k}_2}\right)\right\} + \text{c. c.}, \tag{8}$$

where c.c. denotes the complex conjugate and replacement of the subscripts by $\mathbf{k} \rightleftharpoons \mathbf{k}'$. When there is no field $\mathbf{v}$ a stationary distribution holds with time-independent mean value $M_{\mathbf{k}\mathbf{k}'} \to N_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}')$. In this case grouping of the components in products of the function T with identical subscripts results in δ-function in the frequencies assuring the transformation of (8) into the collision integral [1, 2]. The structure of the last component in the right side of (7) is similar to the structure of the preceding one but its explicit form is not written down because of its awkwardness. The first difference between this component and (8) is the appearance of an integral operator with kernel $R_{\mathbf{k}\mathbf{k}'}$, acting alternately on each of the functions T and Y with symmetrization of these functions with respect to the arguments taken into account. The second difference is the appearance of an integral of R with respect to the time, evaluated between the limits $-\infty < \tau' \le t$.

A qualitative estimate can be presented of the influence of the surface waviness perturbations by the velocity field $\mathbf{v}$ in the case of a finite time t_0 of this action. The component R_2M in the right side of (7) equals zero for $t > t_0$ while the component (8) takes the form of the collision integral [1, 2] for $t \gg t_0$, where this latter contains terms of the form

$$\int d\mathbf{k}_{1\ldots 6}J\int_{-\infty}^{t_0} d\tau R_4(\tau)\exp i\left[(t-\tau)\sum_{n=1}^{4}\omega_n\right]\Phi_s \tag{9}$$

where Φ is the sum of bilinear combinations of matrix elements and the quadruple integral in the wave field momenta estimated for $t/\tau_0 \to \infty$ yields the asymptotic $(\tau_0/t)^8$ for (9). The characteristic constant τ_0 is the greatest value of the integrand in (9) at the boundary points $\mathbf{k}_{1,2}$ of the inertial interval of the wave numbers $\mathbf{k}_1 \le \mathbf{k} \le \mathbf{k}_2$. In the general case, the last component in the right side of (7) describes interaction of four surface waves with the Fourier component of the field **v**, most effective for mutual resonance. This component, additional to the collision integral [1, 2], models the nonlinear mechanism of the nonlocal response, in time, of the system of surface waves to the nonstationary inhomogeneous perturbation. Let us note that if the scale of the homogeneity of the flow considerably exceeds the wavelength then its influence can be taken into account by passage to a moving coordinate system [6]. In such an approximation the correction to the δ correlativity $M_{\mathbf{kk}'}$, which can be substantial for $\mathbf{k} \simeq \mathbf{k}'$, is not taken into account successfully.

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DEFORMATION AND BREAKUP OF A LIQUID FILM UNDER THE ACTION OF THERMOCAPILLARY CONVECTION

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Thermocapillary convection which develops in thin nonisothermal liquid layers produces significant deformation of the free surface [1-5]. This problem was considered in [1] within the framework of a model which neglected capillary pressure. A solution was obtained in [2] for the special case of harmonic temperature distribution; the problem was solved in an approximation linear in temperature perturbation. A more general formulation was considered in [3], where the equation of the free surface was found in an approximation analogous to the boundary layer approximation. The present study will offer new experimental results and define conditions under which thermocapillary convection in a liquid leads to breakup of the film into individual drops.

1. If the plane upon which a thin film of liquid is deposited is oriented perpendicular to the acceleration of gravity then in dimensionless variables the equation of the free surface will have the form [3]

$$\xi^2 + \xi'^2 - 2\xi\xi'' + \varepsilon\vartheta(x) = C, \tag{1.1}$$

where $\xi(x)$ is the local thickness of the liquid layer; $\vartheta(x)$, the temperature of the free surface; C, a constant defined from additional conditions; $\varepsilon = 3\Delta T\sigma_T'/\sigma_0$; σ_0, the mean surface tension coefficient: $\sigma_T' = |d\sigma/dT|$; ΔT, the characteristic temperature difference, for example, between the hotter and colder parts of the layer, per

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